Convergence estimates for the numerical approximation of homoclinic solutions

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Abstract

This article is concerned with the numerical computation of homoclinic solutions converging to a hyperbolic or semi-hyperbolic equilibrium of a system $\dot{u} = f(u, \mu)$. The approximation is done by replacing the original problem by a boundary value problem on a finite interval and introducing an additional phase condition to make the solution unique. Numerical experiments have indicated that the parameter μ is much better approximated than the homoclinic solution. This was proved in Schecter (1995) for phase conditions fulfilling an additional 'niceness' assumption, which is unfortunately not satisfied for the phase condition most commonly used in numerical experiments and which actually suggested the super-convergence result. Here, this result is proved for arbitrary phase conditions. Moreover, it is shown that it is sufficient to approximate the original boundary value problem to first order when considering semi-hyperbolic equilibria extending a result of Schecter (1993).

Keywords. homoclinic bifurcation, convergence, boundary value problem

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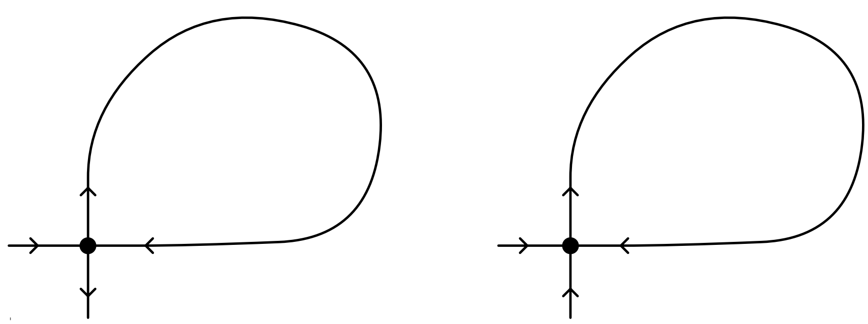
1 Introduction

In recent years several authors have investigated the numerical approximation of homoclinic solutions. The approach most commonly used consists of truncating the infinite interval $\mathbb{R}$ to a finite interval $[T_-, T_+]$ for $T_- < 0 < T_+$ and imposing boundary conditions at the end points $t = T_-$ and $t = T_+$. Admissible boundary conditions may be obtained by requiring the solution to be contained in certain linear or non-linear approximations of the invariant manifolds at the equilibrium towards which the homoclinic solution converges. In order to make the solution of the truncated boundary value problem unique, a phase condition is employed. There are several results available providing estimates of the error made by the truncation and investigating convergence and stability properties of the truncated problem. In case the underlying equilibrium is hyperbolic, these questions are studied in Beyn (1990*b*) and Friedman & Doedel (1991). The case of a semi-hyperbolic equilibrium is investigated in Schecter (1993), see also Friedman & Doedel (1993), Friedman (1993), Bai & Champneys (1994) or Canale (1994). There are also codes available in which these algorithms have been implemented, see for example Champneys, Kuznetsov & Sandstede (1995*b*, 1995*a*) for a driver based on the software package AUTO86 (Doedel (1981)). Another approach for computing homoclinic solutions to hyperbolic equilibria is treated in Moore (1995).

To be specific, consider a system

$$\dot{u} = f(u, \mu) \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R}.$$

Throughout, we shall assume that $f \in C^\infty$, but we remark that $f \in C^4$ (C^5) for the hyperbolic (semi-hyperbolic) case are sufficient. Suppose that $p(\mu)$ is an equilibrium depending on μ and that $q(t)$ is a homoclinic solution to $p(\mu_0)$ for $\mu = \mu_0$, see Figure 1. We assume



(i) Hyperbolic equilibrium (ii) Semi-hyperbolic equilibrium

Figure 1: Homoclinic solutions to hyperbolic (i) or semi-hyperbolic (ii) equilibria

that

$$\sigma(D_u f(p(\mu_0),\mu_0)) \subset (-\infty,-\lambda^s)\cup(\lambda^u,\infty) \qquad \text{for hyperbolic equilibria}$$
$$\sigma(D_u f(p(\mu_0),\mu_0)) \subset (-\infty,-\lambda^s)\cup[0,\infty) \qquad \text{for semi-hyperbolic equilibria}$$

for some constants λ^s, $\lambda^u > 0$. The truncated boundary value problem is given by

$$\begin{pmatrix} \dot{u} - f(u,\mu) \\ J_T(u,\mu) \\ P_0^u(\mu)(u(T_+)-p(\mu)) \\ P_0^s(\mu)(u(T_-)-p(\mu)) \end{pmatrix} = 0, \tag{1.1}$$

for $t \in T = [T_-, T_+]$, see Figure 2. Here P_0^u and P_0^s are spectral projections associated with the above decomposition of the spectrum of $D_u f(p(\mu),\mu)$. J_T denotes the phase condition. The phase condition most commonly used is given by

$$J_T(u,\mu) = \int_{T_-}^{T_+} \langle \dot{q}(t), u(t)-q(t)\rangle \, dt, \tag{1.2}$$

see Friedman & Doedel (1991).

It has been observed numerically in Beyn (1990a) and Friedman & Doedel (1993) using the phase condition given in (1.2) that the parameter μ is actually much better approximated than the homoclinic solution itself. The estimates expected for the error made are

$$\begin{aligned} &|\mu_T-\mu_0| \le C(e^{-(\lambda^u+2\lambda^s)T_+} + e^{(\lambda^s+2\lambda^u)T_-}) && \text{for hyperbolic equilibria} \\ &|\mu_T-\mu_0| \le C(e^{\lambda^s T_-} + e^{-2\lambda^s T_+}) && \text{for semi-hyperbolic equilibria,} \end{aligned} \tag{1.3}$$

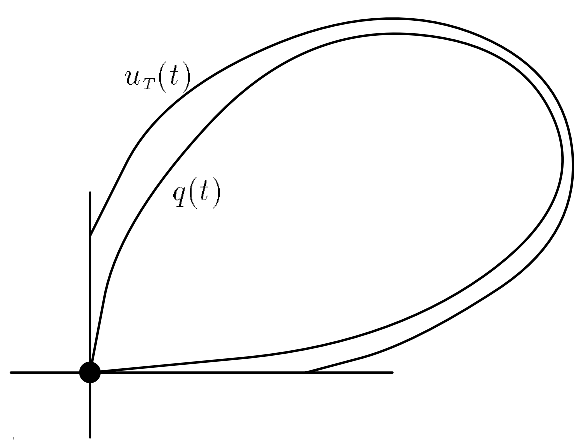


Figure 2: The truncated boundary value problem with the homoclinic solution $q(t)$ and the approximation $u_T(t)$

where (u_T, μ_T) is the solution of (1.1). In Beyn (1990a), a weaker estimate for the hyperbolic case is proved. The inequalities (1.3) have been proved in Schecter (1995) under the hypotheses that the phase condition fulfills an additional 'niceness' assumption and that the center-unstable manifold is approximated to at least quadratic order in case the equilibrium is semi-hyperbolic. Note that the phase condition (1.2) is not 'nice'.

In this article, we shall prove the super-convergence estimates (1.3) for general phase conditions. Moreover, we extend the results obtained in Schecter (1993) concerning convergence and stability of the truncated boundary value problem to linear approximations of the center-unstable manifolds in the case of semi-hyperbolic equilibria.

The proof of the super-convergence result is based on Lin's method Lin (1990). We use here a version investigated in Sandstede (1993). In contrast to Lin (1990) and Schecter (1995), we are going to parametrize solutions of (1.1) not according to

$$u(t) = q(t) + v(t)$$

but according to

$$u^{\pm}(t) = q^{\pm}(b^{\pm}, \mu)(t) + v^{\pm}(t) \text{ for } t > 0 \text{ or } t < 0.$$

Here, the functions $q^{+}(b^{+}, \mu)$ and $q^{-}(b^{-}, \mu)$ parametrize the stable and unstable (or center-unstable) manifolds in a neighborhood of $q(0)$. This allows us to obtain the sharp estimate (1.3) as all obstructing terms appear to drop out during the calculations. It is straightforward to extend the results presented here to different boundary conditions with some obvious changes in the Lemmata 5.2 and 6.4 as well as in the main results.

In order to prove that (1.1) is well-posed even for semi-hyperbolic equilibria, we essentially use the results obtained in Schecter (1993) but employ a sharper version of Banach's fixed point theorem which is better adapted to the present situation.

The remainder of the paper is organized as follows. In sections 2 and 3 the main results for hyperbolic and semi-hyperbolic equilibria, respectively, are stated. Section 4 contains some lemmata which are needed for the proofs. Finally, the main results are proved in section 5 and 6 separately for the hyperbolic and semi-hyperbolic case. In the last section, numerical simulations are presented confirming the theoretical results obtained.

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2 The hyperbolic case (results)

Consider the differential equation

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R} \tag{2.1}$$

for $f \in C^\infty$. Throughout, we assume that p_0 is a hyperbolic equilibrium for $\mu = \mu_0$. Hence, there exist numbers λ^s, $\lambda^u > 0$ such that

$$\sigma(D_u f(p_0, \mu_0)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda < -\lambda^s \text{ or } \mathrm{Re}\lambda > \lambda^u\}. \tag{2.2}$$

Moreover, let $q(t)$ be a homoclinic solution of (2.1) for $\mu = \mu_0$ converging to p_0 for $t \to \pm\infty$. We assume that the following non-degeneracy condition is satisfied:

(H1) $T_{q(0)} W^u(p_0, \mu_0) \cap T_{q(0)} W^s(p_0, \mu_0) = \mathbb{R}\dot{q}(0)$.

Owing to this hypothesis, there exists a unique (up to constant multiples) bounded solution $\psi(t)$ of the adjoint variational equation

$$\dot{w} = -D_u f(q(t), \mu_0)^* w.$$

We assume that the Melnikov integral associated with $q(t)$ does not vanish:

(H2) $\int_{-\infty}^{\infty} \langle \psi(t), D_\mu f(q(t), \mu_0) \rangle \, dt =: M \neq 0$.

Due to the hyperbolicity of p_0, there exists a family $p(\mu)$ of equilibria of (2.1) for μ close to μ_0. Moreover, (2.2) still holds for the spectrum of the linearization at $(p(\mu), \mu)$ provided $|\mu - \mu_0|$ is sufficiently small. We denote the spectral projections associated with the stable and unstable parts of $\sigma(D_u f(p(\mu), \mu))$ by $P_0^s(\mu)$ and $P_0^u(\mu)$, respectively.

Next, we consider the numerical approximation of $q(t)$ obtained by solving a truncated problem. To this end, choose an interval $T := [T_-, T_+] \subset \mathbb{R}$ for $T_- < 0 < T_+$ and $|T_-|$, T_+ large and denote the homoclinic solution $q(t)$ restricted to the interval $[T_-, T_+]$ by $q_T = q|_{[T_-, T_+]}$. Then Beyn (1990*b*) introduced the following boundary value problem defined for $u \in C^1(T, \mathbb{R}^n)$ and $\mu \in \mathbb{R}$:

$$\begin{pmatrix} \dot{u} - f(u, \mu) \\ J_T(u, \mu) \\ P_0^u(\mu)(u(T_+) - p(\mu)) \\ P_0^s(\mu)(u(T_-) - p(\mu)) \end{pmatrix} = 0 \tag{2.3}$$

where $t \in T$. Here, $J_T(u, \mu)$ denotes a phase condition satisfying the following hypothesis

(P) $J_T : C^1(T, \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}$ is of class C^1 and $D_u J_T(q_T, \mu_0)\dot{q}_T \geq d_0 > 0$ for all $|T_-|$, T_+ sufficiently large. Moreover, $J_T(q_T, \mu_0) \to 0$ as $|T_-|$, $T_+ \to \infty$.

The main result is the following theorem.

Theorem 1 *Assume that (H1), (H2) and (P) are fulfilled. Then the unique solution (u_T, μ_T) of (2.3) known to exist by Beyn (1990*b*) satisfies*

$$|\mu_0 - \mu_T| \leq C(e^{-(\lambda^u+2\lambda^s)T_+} + e^{(\lambda^s+2\lambda^u)T_-}),$$

where C is a constant independent of T_- and T_+.

3 The semi-hyperbolic case (results)

Consider the equation

$$\dot{u} = f(u, \mu), \qquad (u, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \tag{3.1}$$

for $f \in C^\infty$ with $m = 1$. Assume that (3.1) possesses a smooth family $p(\mu)$ of equilibria for μ close to μ_0 such that $p(\mu_0) = p_0$. Suppose that the spectrum of the linearization at $p(\mu)$ is given by

$$\sigma(D_u f(p(\mu), \mu)) \subset \{\lambda \in \mathbb{C} \,|\, \mathrm{Re}\lambda < -\lambda^s \text{ or } \mathrm{Re}\lambda > \lambda^u\} \cup \{0\} \tag{3.2}$$

for some numbers λ^s, $\lambda^u > 0$. Moreover, let zero be a simple eigenvalue of $D_u f(p(\mu), \mu)$ for all μ. The vector field on the one-dimensional center manifold should be non-degenerate, that is the family $p(\mu)$ consists of non-degenerate saddle-node equilibria:

(S1) The vector field on $W^c(p_0, \mu_0)$ is given by

$$\dot{u} = a(u - p_0)^2 + O(|u - p_0|^2)$$

for some $a \neq 0$ and $u \in W^c(p_0, \mu_0)$.

Next, we assume that a homoclinic solution exists for $\mu = \mu_0$:

(S2) There exists a homoclinic solution $q(t) \in W^{cu}(p_0, \mu_0) \cap W^s(p_0, \mu_0)$ converging to p_0 and $q(t) \notin W^u(p_0, \mu_0)$.

Again, we assume that the vector field is non-degenerate for $\mu = \mu_0$ and with respect to perturbations in μ, that is:

(S3) $T_{q(0)}W^{cu}(p_0,\mu_0) \cap T_{q(0)}W^s(p_0,\mu_0) = \mathbb{R}\dot{q}(0)$.

(S4) The extended manifolds $\tilde{W}^{cu}(p_0,\mu_0)$ and $\tilde{W}^s(p_0,\mu_0)$ of the system

$$\begin{aligned} \dot{u} &= f(u,\mu) \\ \dot{\mu} &= 0 \end{aligned}$$

intersect transversely along $q(t)$.

We consider the following boundary value problem

$$\hat{F}(u,\mu) := \begin{pmatrix} \dot{u} - f(u,\mu) \\ J_T(u,\mu) \\ P_0^{cu}(\mu)(u(T_+) - p(\mu)) \\ P_0^s(\mu)(u(T_-) - p(\mu)) \end{pmatrix} = 0 \tag{3.3}$$

for $u \in C^1([T_-,T_+],\mathbb{R}^n)$. The phase condition J_T is assumed to satisfy hypothesis (P) stated in the previous section. Again, we will denote the homoclinic solution $q(t)$ restricted to the interval $[T_-,T_+]$ by q_T. Then we have the following theorem.

Theorem 2 *There exists a $\delta > 0$ such that for any T_+, $|T_-| > 1/\delta$ there is a unique solution (u_T,μ_T) of (3.3) in $B_{\delta/T_+}(q_T,\mu_0) \subset C^1([T_-,T_+],\mathbb{R}^n) \times \mathbb{R}$. Moreover, the estimates*

$$\begin{aligned} |u_T - q_T| &\leq C(1/|T_-|^2 + e^{-2\lambda^s T_+}) \\ |\mu_T - \mu_0| &\leq C(e^{\lambda^s T_-} + e^{-2\lambda^s T_+}) \end{aligned}$$

hold.

The above theorem shows that there is a unique solution of the truncated boundary value problem in a ball of size δ/T_+ independent of T_-. Moreover, there is a super-convergence in the parameter μ without any further assumptions on the phase condition.

Next, we consider the case $\mu \in \mathbb{R}^2$ such that the assumptions (S1) to (S4) are fulfilled with respect to μ_2 for $\mu_1 = 0$. We shall investigate the boundary value problem

$$F(u,\mu) := \begin{pmatrix} \dot{u} - f(u,\mu) \\ J_T(u,\mu) \\ P_0^{cu}(\mu)(u(T_+) - p(\mu)) \\ P_0^s(\mu)(u(T_-) - p(\mu)) \\ \mu_1 \end{pmatrix} = 0 \tag{3.4}$$

for $u \in C^1([T_-,T_+],\mathbb{R}^n)$. Then we have the following theorem.

Theorem 3 *There exists a $\delta > 0$ such that for any $|T_-| > T_+ > 1/\delta$ the following holds:*

(i) The estimate

$$|DF^{-1}(u,\mu)| \leq C|T_-|$$

holds for all $(u,\mu) \in B_{\delta/|T_-|}(q_T, \mu_0)$.

(ii) Suppose that $|T_-| \geq T_+ \geq C + \frac{1}{2\lambda^s} \ln|T_-|$ for some constant C. Then the Chord method

$$x_{n+1} = x_n - DF(q_T, \mu_T)^{-1} F(x_n)$$

associated with F converges in $B_{\delta/|T_-|}(q_T, \mu_0)$ to the unique solution described in Theorem 2.

Note that the error estimates provided in Theorem 2 are valid for equation (3.4), too.

Thus, in a ball of size $\delta/|T_-|$, the stability result (i) holds. However, convergence of the Chord method requires an additional - but slightly improved - relation on T_- and T_+ as in Schecter (1993). One of the improvements compared to Schecter (1993) is that we allow for a linear approximation of the center-unstable manifold instead of requiring a nonlinear approximation including second order terms for the convergence result.

4 Two lemmata

Consider two manifolds N^+ and N^- embedded in $\mathbb{R}^n$ with dimensions $\dim N^+ = n_+$ and $\dim N^- = n_-$ such that $n_+ + n_- = n$. Assume that they intersect along a one-dimensional manifold Q near a point $q \in Q$ such that

$$(T_qN^+ + T_qN^-) \oplus Z = \mathbb{R}^n \tag{4.1}$$

for some one-dimensional space Z. Then we can decompose the tangent spaces at q according to

$$\begin{aligned} T_qN^+ &= T_qQ \oplus Y^+ \\ T_qN^- &= T_qQ \oplus Y^- \end{aligned} \tag{4.2}$$

and

$$T_qQ \oplus Z \oplus Y^+ \oplus Y^- = \mathbb{R}^n \tag{4.3}$$

holds. Throughout, we will denote the projection onto X with kernel Y by $P(X,Y)$. Then define the projections

$$\begin{aligned} P_q &= P\Big(T_qQ, Z \oplus Y^+ \oplus Y^-\Big) \\ P_\psi &= P\Big(Z, T_qQ \oplus Y^+ \oplus Y^-\Big) \\ P_Y &= P\Big(Y^+ \oplus Y^-, Z \oplus T_qQ\Big) \end{aligned} \tag{4.4}$$

according to the decomposition (4.3). We choose a non-zero vector $\dot{q} \in T_qQ$ – which is not a time-derivative at the moment but just a sloppy notation – such that $T_qQ = \mathbb{R}\dot{q}$ and parametrize the intersection of N^+ and N^- locally near the point q by a function

$$\begin{aligned} \alpha &\mapsto q + \alpha\dot{q} + h(\alpha) \\ h(\alpha) &= (h_\psi(\alpha), h_+(\alpha), h_-(\alpha)) \in Z \oplus Y^+ \oplus Y^- \end{aligned} \tag{4.5}$$

with $h(\alpha) = O(\alpha^2)$. Now suppose that the manifolds $N^+ = N^+(\mu)$ and $N^- = N^-(\mu)$ depend smoothly on a parameter $\mu \in \mathbb{R}$ such that the above configuration arises for $\mu = \mu_0$. Then there are unique functions

$$H^+ : T_qN^+(\mu_0) \times \mathbb{R} \to \mathbb{R}^n \qquad H^- : T_qN^-(\mu_0) \times \mathbb{R} \to \mathbb{R}^n$$

such that

$$\begin{aligned} &H^+(x^+, \mu) \in N^+(\mu) && H^+(x^+, \mu) - q - x^+ \in Z \oplus Y^- \\ &H^-(x^-, \mu) \in N^-(\mu) && H^-(x^-, \mu) - q - x^- \in Z \oplus Y^+. \end{aligned} \tag{4.6}$$

Using the parametrization (4.5) of Q locally near q, we define

$$\begin{aligned} q^+(\alpha, b^+, \mu) &:= H^+(\alpha + h(\alpha) + b^+, \mu) \\ q^-(\alpha, b^-, \mu) &:= H^-(\alpha + h(\alpha) + b^-, \mu) \end{aligned} \tag{4.7}$$

for $\alpha \in \mathbb{R}$, $b^+ \in Y^+$ and $b^- \in Y^-$. Some properties of these functions are collected in the next lemma.

Lemma 4.1 *The functions q^+ and q^- satisfy*

(i) $q^+(\alpha, b^+, \mu) - q^-(\alpha, b^-, \mu) = b^+ - b^- + R_1(\alpha, b^+, b^-, \mu)$

(ii) $P_q\Big(q^+(\alpha, b^+, \mu) - q^-(\alpha, b^-, \mu)\Big) = 0$

(iii) $P_\psi\Big(q^+(\alpha, b^+, \mu) - q^-(\alpha, b^-, \mu)\Big) = \tilde{M}(\mu - \mu_0) + R_2(\alpha, b^+, b^-, \mu),$

where

$$\begin{aligned} R_1 = R_1(\alpha, b^+, b^-, \mu) &= O\Big(|\mu - \mu_0| + (|b^+| + |b^-|)(|b^+| + |b^-| + |\alpha|)\Big) \\ R_2 = R_2(\alpha, b^+, b^-, \mu) &= O\Big((|b^+| + |b^-| + |\mu - \mu_0|)(|b^+| + |b^-| + |\alpha| + |\mu - \mu_0|)\Big). \end{aligned}$$

Moreover, $\tilde{M}$ is non-zero if and only if the extended manifolds $(N^+(\mu), \mu)$ and $(N^-(\mu), \mu)$ intersect transversely in $\mathbb{R}^n \times \mathbb{R}$ in the point q.

Proof. Using (4.6) and (4.7), we obtain

$$
\begin{aligned}
q^+(\alpha, b^+, \mu) &= q + \alpha\dot{q} + h_+(\alpha) + b^+ + q_R^+(\alpha, b^+, \mu) \\
q^-(\alpha, b^-, \mu) &= q + \alpha\dot{q} + h_-(\alpha) + b^- + q_R^-(\alpha, b^-, \mu)
\end{aligned}
\tag{4.8}
$$

for some functions $q_R^+(\alpha, b^+, \mu) \in Z \oplus Y^-$ and $q_R^-(\alpha, b^-, \mu) \in Z \oplus Y^+$. Therefore,

$$
q^+(\alpha, b^+, \mu) - q^-(\alpha, b^-, \mu) = h_+(\alpha) - h_-(\alpha) + b^+ - b^- + q_R^+(\alpha, b^+, \mu) - q_R^-(\alpha, b^-, \mu) \tag{4.9}
$$

and (ii) follows by applying the projection P_q to the identity (4.9). In order to show (i), we set $\mu = \mu_0$. Then, $q^+(\cdot, \cdot, \mu_0)$ describe N^+ locally near q as a graph over its tangent space $T_q Q \oplus Y^+$. In particular, $q_R^+(\alpha, b^+, \mu_0) = O((|\alpha| + |b^+|)^2)$ and similarly $q_R^-(\alpha, b^-, \mu_0) = O((|\alpha| + |b^-|)^2)$. Therefore, using (4.5), we obtain

$$
h_+(\alpha) - h_-(\alpha) + q_R^+(\alpha, b^+, \mu_0) - q_R^-(\alpha, b^-, \mu_0) = O((|\alpha| + |b^+| + |b^-|)^2).
$$

On the other hand, both $q^+(\alpha, 0, \mu_0)$ and $q^-(\alpha, 0, \mu_0)$ parametrize $N^+ \cap N^-$ for $\mu = \mu_0$, whence

$$
q^+(\alpha, 0, \mu_0) = q + \alpha\dot{q} + h(\alpha) = q^-(\alpha, 0, \mu_0).
$$

Hence, we conclude that the difference

$$
h_+(\alpha) - h_-(\alpha) + q_R^+(\alpha, b^+, \mu_0) - q_R^-(\alpha, b^-, \mu_0) = O((|b^+| + |b^-|)(|\alpha| + |b^+| + |b^-|))
$$

is actually of order $O((|b^+| + |b^-|)(|\alpha| + |b^+| + |b^-|))$. Subsuming the dependence on μ into the $O(|\mu - \mu_0|)$ term, (i) obtains. Finally, we have to prove (iii), which is now an easy consequence of the estimates obtained above. The statement about $\tilde{M}$ is obvious. $\square$

Next, we provide a lemma which is needed later to match two pieces of solutions together. It mainly uses the abstract estimates obtained in the previous lemma.

Lemma 4.2 *Assume that in the setting described above the constant $\tilde{M}$ appearing in Lemma 4.1 is non-zero. Suppose there are two functions*

$$
\begin{array}{rcccc}
w^+ & : & \mathbb{R} \times Y^+ \times \mathbb{R} \times \mathbb{R}^2 & \to & Y^- \times Z \\
 & & (\alpha, b^+, \mu, \nu) & \mapsto & w^+(\alpha, b^+, \mu, \nu) \\
w^- & : & \mathbb{R} \times Y^- \times \mathbb{R} \times \mathbb{R}^2 & \to & Y^+ \times Z \\
 & & (\alpha, b^-, \mu, \nu) & \mapsto & w^-(\alpha, b^-, \mu, \nu)
\end{array}
$$

smooth in (α, b^+, μ) and (α, b^-, μ), respectively, such that the norms of w^+ and w^- and their first derivatives with respect to $(\alpha, b^\pm, \mu)$ are smaller than some function $d(\nu)$ satisfying

$0 < d(\nu) < \epsilon$ for all $(\alpha, b^+, b^-, \mu) \in B_\delta(0,0,0,\mu_0)$ and all ν. Then for ϵ sufficiently small there exist unique functions b_*^+, b_*^-, μ_* depending on (α, ν) and being smooth in α such that

$$q^+(\alpha, b_*^+, \mu_*) + w^+(\alpha, b_*^+, \mu_*) = q^-(\alpha, b_*^-, \mu_*) + w^-(\alpha, b_*^-, \mu_*) \tag{4.10}$$

holds for all (α, ν). Moreover, the estimate

$$|b_*^+| + |b_*^-| + |\mu_* - \mu_0| \le K d(\nu) \tag{4.11}$$

holds uniformly in α as well as for the derivatives with respect to α for some constant K.

Proof. We will use the abbreviations $c^\pm := (\alpha, b^\pm, \mu)$. In order to solve equation (4.10), we project it into the complementary subspaces defined in (4.3). Owing to Lemma 4.1 (ii) and $w^+ \in Y^- \times Z$, $w^- \in Y^+ \times Z$, the projection of (4.10) onto T_qQ along $Z \oplus Y^+ \oplus Y^-$ vanishes identically. Hence, it remains to consider the projections onto Z and $Y^+ \oplus Y^-$ given by

$$\begin{aligned} P_Y\Big(q^+(c^+) - q^-(c^-) + w^+(c^+,\nu) - w^-(c^-,\nu)\Big) &= 0 \\ P_\psi\Big(q^+(c^+) - q^-(c^-) + w^+(c^+,\nu) - w^-(c^-,\nu)\Big) &= 0, \end{aligned}$$

where we used definition (4.4). Substituting the expansions obtained in Lemma 4.1 yields

$$G(\mu, \alpha, b^+, b^-, \nu) = 0$$

with

$$G(\mu, \alpha, b^+, b^-, \nu) := \begin{pmatrix} b^+ - b^- + R_1(\alpha, b^+, b^-, \mu) + P_Y(w^+(c^+,\nu) - w^-(c^-,\nu)) \\ \tilde{M}(\mu - \mu_0) + R_2(\alpha, b^+, b^-, \mu) + P_\psi(w^+(c^+,\nu) - w^-(c^-,\nu)) \end{pmatrix}, \tag{4.12}$$

where

$$\begin{aligned} R_1 = R_1(\alpha, b^+, b^-, \mu) &= O\Big(|\mu - \mu_0| + (|b^+| + |b^-|)(|b^+| + |b^-| + |\alpha|)\Big) \\ R_2 = R_2(\alpha, b^+, b^-, \mu) &= O\Big((|b^+| + |b^-| + |\mu - \mu_0|)(|b^+| + |b^-| + |\alpha| + |\mu - \mu_0|)\Big). \end{aligned} \tag{4.13}$$

is the remainder term appearing in Lemma 4.1.

By the assumptions on $w^\pm$ and $\tilde{M}$, we see that $D_{(\mu,b^+,b^-)}G(\mu, \alpha, b^+, b^-, \nu)$ is invertible for all $(\mu, \alpha, b^+, b^-) \in B_\delta(\mu_0, 0, 0, 0)$ and all $\nu \in \mathbb{R}^2$. In addition,

$$\left| \Big(D_{(\mu,b^+,b^-)}G(\mu, \alpha, b^+, b^-, \nu)\Big)^{-1} \right| \le C$$

and for any $\eta > 0$ there exist $\delta, \epsilon > 0$ such that

$$\left| D_{(\mu,b^+,b^-)}G(\mu, \alpha, b^+, b^-, \nu) - D_{(\mu,b^+,b^-)}G(\mu_0, 0, 0, 0, \nu) \right| \le \tilde{\eta}$$

holds for all $(\mu, \alpha, b^+, b^-) \in B_\delta(\mu_0, 0, 0, 0)$ and $\nu \in \mathbb{R}^2$. Moreover, we have

$$|G(\mu_0, 0, 0, 0, \nu)| \leq d(\nu) < \epsilon$$

by assumption. Hence, for ϵ sufficiently small, there exists a unique family $(\mu_*, b_*^+, b_*^-)(\alpha, \nu)$ of solutions of (4.10) depending smoothly on (α, ν) by invoking an implicit function theorem. The estimate for $|b_*^+| + |b_*^-| + |\mu_* - \mu_0|$ follows now directly from (4.12) and (4.13). Note that this estimate holds for the derivatives of (μ_*, b_*^+, b_*^-) with respect to α, too. □

5 The hyperbolic case (proofs)

First, we shall apply Lemma 4.1 to the manifolds $N^+(\mu) = W^s(p(\mu), \mu)$ and $N^-(\mu) = W^-(p(\mu), \mu)$. Let $q := q(0)$ and choose Q to be the homoclinic solution $q(t)$ and $Z = \mathbb{R}\psi(0)$. Owing to hypothesis (H1), the assumptions of section 4 are satisfied. Next, we use the information about the dynamics nearby the stable and unstable manifolds. In the following, we consider therefore the solutions

$$\begin{aligned} &q^+(\alpha, b^+, \mu)(t) \in W^s(p(\mu), \mu) \qquad && q^+(\alpha, b^+, \mu)(0) = q^+(\alpha, b^+, \mu) \qquad && \text{for } t \geq 0 \\ &q^-(\alpha, b^-, \mu)(t) \in W^u(p(\mu), \mu) \qquad && q^-(\alpha, b^-, \mu)(0) = q^-(\alpha, b^-, \mu) \qquad && \text{for } t \leq 0. \end{aligned}$$

As an abbreviation, we set

$$c^\pm = (\alpha, b^\pm, \mu).$$

The variational equations along these solutions

$$\dot{v} = D_u f(q^\pm(\alpha, b^\pm, \mu)(t), \mu)\, v = D_u f(q^\pm(c^\pm)(t), \mu)\, v \tag{5.1}$$

admit exponential dichotomies for $t \geq 0$ and $t \leq 0$, respectively. That is, there exist projections $P_+^s(c^+)(t)$ and $P_-^u(c^-)(t)$ defined for $t \geq 0$ and $t \leq 0$, respectively, such that

$$\begin{aligned} |\Phi_+(c^+, t, s) P_+^s(c^+)(s)| &\leq K e^{-\lambda^s(t-s)} \qquad && t \geq s \geq 0 \\ |\Phi_+(c^+, s, t)(1 - P_+^s(c^+)(t))| &\leq K e^{-\lambda^u(t-s)} \qquad && t \geq s \geq 0 \\ |\Phi_-(c^-, t, s)(1 - P_-^u(c^-)(s))| &\leq K e^{-\lambda^s(t-s)} \qquad && s \leq t \leq 0 \\ |\Phi_-(c^-, s, t) P_-^u(c^-)(t)| &\leq K e^{-\lambda^u(t-s)} \qquad && s \leq t \leq 0 \end{aligned} \tag{5.2}$$

holds. Here, $\Phi_\pm(c^\pm, t, s)$ denotes the evolution of (5.1) for either $s, t \geq 0$ or $s, t \leq 0$. Denoting the complementary projections by

$$\begin{aligned} P_+^u(c^+)(t) &:= 1 - P_+^s(c^+)(t) \qquad && \text{for } t \geq 0 \\ P_-^s(c^-)(t) &:= 1 - P_-^u(c^-)(t) \qquad && \text{for } t \leq 0, \end{aligned}$$

we have in addition that the subspaces

$$\begin{aligned}\ker P^s_+(c^+)(0) = \mathrm{R}P^u_+(c^+)(0) &= \mathbb{R}\psi(0) \oplus Y^+ \\ \ker P^u_-(c^-)(0) = \mathrm{R}P^s_-(c^-)(0) &= \mathbb{R}\psi(0) \oplus Y^-\end{aligned} \tag{5.3}$$

do not depend on c^+ or c^-. We will use the abbreviations

$$\begin{array}{ll}\Phi_+(c^+,t,s)P^s_+(c^+)(s) =: \Phi^s_+(c^+,t,s) & \Phi_+(c^+,s,t)P^u_+(c^+)(t) =: \Phi^u_+(c^+,s,t) \\ \Phi_-(c^-,s,t)P^u_-(c^-)(t) =: \Phi^u_-(c^-,s,t) & \Phi_-(c^-,t,s)P^s_-(c^-)(s) =: \Phi^s_-(c^-,t,s).\end{array}$$

The estimates in (5.2) are true for the derivatives with respect to c^+ and c^-, too, see Sandstede (1993, Lemma 1.1) for the proofs.

We shall use the solutions $q^+(c^+)$ and $q^-(c^-)$ as a parametrization which respect to which solutions of the boundary value problem (2.3) are constructed. Hence, define

$$\begin{array}{ll}u^+(t) = q^+(c^+)(t) + v^+(t) & t \geq 0 \\ u^-(t) = q^-(c^-)(t) + v^-(t) & t \leq 0.\end{array} \tag{5.4}$$

As we can prescribe the stable component of u^+ by choosing c^+ (and similarly for the unstable component of u^-), we shall require that

$$v^+(0) \in \mathbb{R}\psi(0) \oplus Y^+ \qquad v^-(0) \in \mathbb{R}\psi(0) \oplus Y^-. \tag{5.5}$$

Observe that these spaces are precisely the subspaces occurring in (5.3) as the kernels of the projections which are in fact independent of c^+ and c^-. The functions $u^\pm$ solve the equation

$$\dot{u} = f(u, \mu)$$

if and only if the functions $v^\pm$ solve

$$\dot{v}^\pm = D_u f(q^\pm(c^\pm)(t), \mu)\, v^\pm + g^\pm(t, v^\pm, c^\pm) \tag{5.6}$$

for $t \geq 0$ or $t \leq 0$, respectively. Here, the nonlinearities are given by

$$\begin{aligned}g^\pm(t, v, c^\pm, \mu) &= f(q^\pm(c^\pm)(t) + v, \mu) - f(q^\pm(c^\pm)(t), \mu) - D_u f(q^\pm(c^\pm)(t), \mu)v \\ &= O(|v|^2).\end{aligned} \tag{5.7}$$

The estimate for the nonlinearity is valid for derivatives with respect to $c^\pm = (\alpha, b^\pm, \mu)$, too.

We are going to rewrite the differential equation (5.6) on the interval $T = [T_-, T_+]$ together with the condition (5.5) as an integral equation according to

$$\begin{aligned}v^+(t) &= \Phi^u_+(t, T_+)a^+ + \textstyle\int_{T_+}^t \Phi^u_+(t,s)g^+(t, v^+(s), c^+)\, ds + \\ &\quad \textstyle\int_0^t \Phi^s_+(t,s)g^+(t, v^+(s), c^+)\, ds \\ v^-(t) &= \Phi^s_-(t, T_-)a^- + \textstyle\int_{T_-}^t \Phi^s_-(t,s)g^-(t, v^-(s), c^-)\, ds + \\ &\quad \textstyle\int_0^t \Phi^u_-(t,s)g^-(t, v^-(s), c^-)\, ds\end{aligned} \tag{5.8}$$

where $a^+ \in E_0^u$ and $a^- \in E_0^s$. Using the estimates (5.2) together with the property (5.3) we shall see that solutions of (5.8) solve the differential equation (5.6) on the interval $T = [T_-, T_+]$, are bounded uniformly in T_+ and T_-, and fulfill equation (5.5). We solve (5.8) in function spaces endowed with exponentially weighted norms. Define

$$\begin{aligned}\|v\|_+ &:= \sup_{t\in[0,T_+]} e^{\lambda^u(T_+-t)}|v(t)| \\ \|v\|_- &:= \sup_{t\in[T_-,0]} e^{\lambda^s(t-T_-)}|v(t)|\end{aligned}$$

and let

$$\begin{aligned}V^+ &:= \{v \in C^0([0,T_+],\mathbb{R}^n) \,|\, \|v\|_+ < \infty\} \\ V^- &:= \{v \in C^0([T_-,0],\mathbb{R}^n) \,|\, \|v\|_- < \infty\}\end{aligned}$$

be spaces equipped with the above defined norms. Let $B_\rho^\pm$ denote the ball of radius ρ around zero in the spaces $V^\pm$.

Lemma 5.1 *There exist $\delta, \rho > 0$ such that for any T_+, $|T_-| > 1/\delta$, any given $(a^+, a^-) \in E_0^u \times E_0^s$ with $|a^+| + |a^-| < \delta$, and any $(\alpha, b^+, b^-, \mu) \in \mathbb{R} \times Y^+ \times Y^- \times \mathbb{R}$ with $|\alpha| + |b^+| + |b^-| + |\mu - \mu_0| < \delta$, there exist unique solutions $v^\pm(a^\pm, c^\pm) \in B_\rho^\pm$ of equation (5.8). Moreover, $v^\pm$ depends smoothly on $(a^\pm, c^\pm)$ and the estimate*

$$\|v^\pm\|_\pm \le C|a^\pm|$$

holds for some constant C depending only on δ. The estimate is valid for derivatives with respect to $c^\pm$, too.

Proof. It is easy to see that the right-hand side of (5.8) defines a smooth mapping from $V^\pm$ into itself using the estimate (5.7) for the nonlinearity $g^\pm$, see Lin (1990), Schecter (1995) or Sandstede (1993) for the details. Note that (5.7) is uniform in $c^\pm$. Hence, we can apply the implicit function theorem to obtain the existence part of the lemma. The estimate given in the lemma follows immediately from the integral equation and g being quadratic in v. $\square$

Next, we consider the boundary conditions at $t = T_+$ and $t = T_-$, respectively, given by

(5.9)
$$\begin{aligned}&P_0^u(\mu)(u(T_+) - p(\mu)) \\ &= P_0^u(\mu)\Big(q^+(c^+)(T_+) + P_+^u(c^+)(T_+)a^+ \\ &\quad + \int_0^{T_+} \Phi_+^s(T_+, s)g^+(t, v^+(s), c^+)\, ds - p(\mu)\Big) = 0\end{aligned}$$

(5.10)
$$\begin{aligned}&P_0^s(\mu)(u(T_-) - p(\mu)) \\ &= P_0^s(\mu)\Big(q^-(c^-)(T_-) + P_+^s(c^-)(T_-)a^- \\ &\quad + \int_0^{T_-} \Phi_-^u(T_-, s)g^-(t, v^-(s), c^-)\, ds - p(\mu)\Big) = 0\end{aligned}$$

using (5.8) and the parametrization (5.4).

Lemma 5.2 *For any T_+, T_- and $c^\pm$ chosen as in Lemma 5.1, there exists a unique pair $(a^+, a^-) \in E_0^u \times E_0^s$ such that $v^\pm(a^\pm(c^\pm), c^\pm)$ solve the boundary conditions (5.9) and (5.10). Moreover, (a^+, a^-) depends smoothly on (c^+, c^-) and the estimates*

$$(5.11) \qquad \begin{array}{rcccl} |a^+| & \leq & C|P_0^u(\mu)(p(\mu) - q^+(c^+)(T_+))| & \leq & Ce^{-2\lambda^s T_+} \\ |a^-| & \leq & C|P_0^s(\mu)(p(\mu) - q^-(c^-)(T_-))| & \leq & Ce^{2\lambda^u T_-} \end{array}$$

hold for $a^\pm$ as well as for derivatives with respect to $c^\pm$.

Proof. The maps $P_0^u(\mu)P_+^u(c^+)(T_+)|_{E_0^u}$ and $P_0^s(\mu)P_+^s(c^-)(T_-)|_{E_0^s}$ are isomorphisms for all T_+ and $|T_-|$ sufficiently large as $\lim_{t\to\infty} P_+^u(c^+)(t) = P_0^u(\mu)$ and similarly for $P_-^s(c^-)(t)$, see Palmer (1984, Lemma 3.4). Using Lemma 5.1 we can therefore solve (5.9) and (5.10) by the implicit function theorem for any fixed T_+ and T_-. The first part of the estimate (5.11) follows now immediately. The second part of the inequalities is a consequence of the quadratic tangency between $W^u(p(\mu), \mu)$ and its tangent space $\mathrm{R}P_0^u(\mu)$ at $p(\mu)$ (and the analogous property for $W^s(p(\mu), \mu)$). $\square$

Therefore, we obtain two pieces of solutions satisfying the differential equation and the boundary conditions at both end points. In order to obtain a solution of the full problem, both solutions have to coincide at $t = 0$.

Lemma 5.3 *For any choice of T_+ and T_- as in Lemma 5.1, there exist a unique vector $(\alpha_T, b_T^+, b_T^-, \mu_T)$ such that*

$$(5.12) \quad q^+(\alpha_T, b_T^+, \mu_T)(0) + v^+(\alpha_T, b_T^+, \mu_T)(0) = q^-(\alpha_T, b_T^-, \mu_T)(0) + v^-(\alpha_T, b_T^-, \mu_T)(0)$$

and

$$J_T(u_T, \mu_T) = 0$$

hold. Here,

$$u_T(t) = q^\pm(\alpha_T, b_T^\pm, \mu_T)(t) + v^\pm(\alpha_T, b_T^\pm, \mu_T)(t)$$

for $t \geq 0$ and $t \leq 0$, respectively. Moreover, the estimate

$$|b_T^+| + |b_T^-| + |\mu_T - \mu_0| \leq C(e^{-(\lambda^u + 2\lambda^s)T_+} + e^{(\lambda^s + 2\lambda^u)T_-})$$

holds.

Proof. We employ Lemma 4.2 in order to solve equation (5.12) by setting

$$w^+(\alpha, b^+, \mu, \nu) := v^+(\alpha, b^+, \mu, T_+)(0)$$
$$w^-(\alpha, b^-, \mu, \nu) := v^-(\alpha, b^-, \mu, T_-)(0)$$

and $\nu = (T_+, T_-)$. By Lemma 5.2 and hypothesis (H2) – which is equivalent to $\tilde{M} \neq 0$ by standard Melnikov theory, see for example Lin (1990) – we see that all the assumptions of Lemma 4.2 are fulfilled. Hence we obtain functions $(\mu_T, b_T^+, b_T^-)(\alpha, T_+, T_-)$ depending smoothly on α such that (5.12) is fulfilled for $(b^\pm, \mu) = (b_T^\pm(\alpha), \mu_T(\alpha))$. The estimate for $|b_T^+| + |b_T^-| + |\mu_T - \mu_0|$ follows now directly from (4.11), (5.11) and the definition of the weighted norm we used for solving (5.8).

It remains to solve for the phase condition. Note that the function $u_T(\alpha)$ obtained by substituting the function $(\mu_T, b_T^+, b_T^-)(\alpha)$ into $u(c^+, c^-)(t)$ is at least C^1 as it solves the integral equation (5.8) and is continuous owing to the choice of (μ, α, b^+, b^-). Hence, we can evaluate the phase condition J_T along the one-parameter family $u(\alpha)$ to get the equation

$$J_T(u_T(\alpha), \mu_T(\alpha)) = 0, \tag{5.13}$$

we have to solve. By definition (4.7) and (4.5) we have $\frac{d}{d\alpha} q^\pm(\mu_0, 0, 0)(t) = \dot{q}(t)$, whence we can solve (5.13) with respect to α due to assumption (P), because the derivative of (μ_T, b_T^+, b_T^-) with respect to α is small by the above arguments. This proves the lemma. $\square$

By Lemma 5.3 the proof of Theorem 1 is completed.

6 The semi-hyperbolic case (proofs)

We will first prove the two statements of Theorem 3. The proofs follows from the work of Schecter (1993) if the following sharper version of Banach's fixed point theorem is going to be used.

Lemma 6.1 *Suppose that X and Y are Banach spaces and $F : X \to Y$ is a C^1-function. Assume that there exists a linear, bounded and invertible operator $A : X \to Y$ such that*

(i) $|A^{-1}(A - DF(x))| \leq \kappa < 1$ *for all* $x \in B_\rho(x_0)$,

(ii) $|A^{-1}F(x_0)| \leq (1 - \kappa)\rho$.

Then there exists a unique point $x_* \in B_\rho(x_0)$ with $F(x_0) = 0$ and the estimates

$$\begin{aligned} |x_0 - x_*| &\leq (1-\kappa)^{-1} |A^{-1}F(x_0)| \\ |DF(x)^{-1}| &\leq (1+\kappa) |A^{-1}| \end{aligned}$$

hold uniformly in $x \in B_\rho(x_0)$. Only the first assumption (i) is needed for the second inequality.

Proof. First we obtain

$$\begin{aligned} &|x - A^{-1}F(x) - y + A^{-1}F(y)| \\ &\leq |A^{-1}(A(x-y) - F(x) + F(y))| \\ &\leq |\int_0^1 A^{-1}(A - DF(y + \tau(x-y)))\, d\tau\, (x-y)| \\ &\leq \sup_{z \in B_\rho(x_0)} |A^{-1}(A - DF(z))|\, |x-y| \\ &\leq \kappa\, |x-y|. \end{aligned}$$

Thus $T(x) := x - A^{-1}F(x)$ is a contraction and we can apply Banach's fixed point theorem, see Chow & Hale (1982, Thm. 2.1). The second estimate follows by applying the same theorem to the contraction

$$B \mapsto A^{-1} + A^{-1}(A - DF(x))B$$

on the space of linear bounded operators from X to Y with $x_0 = A^{-1}$. □

In the notation of Schecter (1993), the original and the truncated boundary value problem for the homoclinic solution can be written according to

$$\begin{aligned} F:\ & C^1([T_-, T_+], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R} \to C^0([T_-, T_+], \mathbb{R}^n) \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_{cu}} \times \mathbb{R} \times \mathbb{R} \\ & (u, \mu_1, \mu_2) \mapsto (\dot{u} - f(u, \mu), B_-(\mu, u(T_-)), B_+(\mu, u(T_+)), J_T(u, \mu), \mu_1). \end{aligned}$$

and

$$\begin{aligned} \tilde{F}:\ & C^1([T_-, T_+], \mathbb{R}^n) \times \mathbb{R} \times \mathbb{R} \to C^0([T_-, T_+], \mathbb{R}^n) \times \mathbb{R}^{n_s} \times \mathbb{R}^{n_{cu}} \times \mathbb{R} \times \mathbb{R} \\ & (u, \mu_1, \mu_2) \mapsto (\dot{u} - f(u, \mu), P^s(\mu)(u(T_-) - p(\mu)), P^{cu}(\mu)(u(T_+) - p(\mu)), J_T(u, \mu), \mu_1), \end{aligned}$$

respectively. For the following arguments, note that in the proofs of Schecter (1993, Lemmata 2.3, 2.4 and 2.5) no use is made of the assumption that the center-unstable manifold is approximated including the quadratic terms.

Using the assumption $T_+ \leq |T_-|$, the inverse of the linearization of F at (q_T, μ_0) can be estimated by

$$
\begin{aligned}
(6.1) \qquad \begin{pmatrix} |u| \\ |\nu_1| \\ |\nu_2| \end{pmatrix} &\leq C \begin{pmatrix} |T_-|\,|z| + |b_-| + |b_+| + |\zeta| + |\eta| \\ |\eta| \\ |T_-|\,|z| + e^{-\lambda^s T_+}|b_-| + |b_+| + |\zeta| + |\eta| \end{pmatrix} \\
&\leq C \begin{pmatrix} |T_-| & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ |T_-| & e^{-\lambda^s T_+} & 1 & 0 & 1 \end{pmatrix} (z, b_-, b_+, \zeta, \eta)^*
\end{aligned}
$$

where

$$(u, \nu_1, \nu_2) = DF(q_T, \mu_0)^{-1}\,(z, b_-, b_+, \zeta, \eta),$$

see Schecter (1993, Lemma 2.4). Here and in the sequel, estimates for vectors like the one in (6.1) are always understood to be component-wise. Using the above definitions of F and $\tilde{F}$, we obtain

$$
(6.2) \qquad |DF(q_T, \mu_0) - D\tilde{F}(q_T, \mu_0)| \leq C \begin{pmatrix} 0 & 0 & 0 \\ |T_-|^{-1} & |T_-|^{-1} & |T_-|^{-1} \\ e^{-\lambda^s T_+} & e^{-\lambda^s T_+} & e^{-\lambda^s T_+} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Furthermore, owing to Schecter (1993, Lemma 2.5 and 2.3), we obtain

$$
(6.3) \qquad \tilde{F}(q_T, \mu_0) \leq C \begin{pmatrix} 0 \\ |T_-|^{-2} \\ e^{-2\lambda^s T_+} \\ 0 \\ 0 \end{pmatrix},
$$

where each component is estimated separately, and

$$
(6.4) \qquad |D\tilde{F}(q_T, \mu_0) - D\tilde{F}(u, \mu)| \leq C\rho
$$

for all $(u, \mu) \in B_\rho(q_T, \mu_0)$ uniformly in $|T_-|$ and T_+.

We shall verify the assumptions of Lemma 6.1 with $A := DF(q_T, \mu_0)$. Using (6.1), (6.2) and (6.4), we have

$$
\begin{aligned}
(6.5) \qquad & |A^{-1}(DF(q_T, \mu_0) - D\tilde{F}(u, \mu))| \\
\leq\ & |A^{-1}(DF(q_T, \mu_0) - D\tilde{F}(q_T, \mu_0))| + |A^{-1}||D\tilde{F}(q_T, \mu_0) - D\tilde{F}(u, \mu)| \\
\leq\ & C(|T_-|^{-1} + e^{-\lambda^s T_+}) + C|T_-|\rho \leq C(|T_-|^{-1} + e^{-\lambda^s T_+} + |T_-|\rho)
\end{aligned}
$$

for all $(u,\mu) \in B_\rho(q_T,\mu_0)$. Therefore, setting

$$\kappa := 1/2 \text{ and } \rho := \frac{1}{4C}|T_-|^{-1},$$

and choosing $|T_-|$, T_+ sufficiently large, hypothesis (i) of Lemma 6.1 is satisfied. This already proves part (i) of Theorem 3 by invoking Lemma 6.1. In order to verify hypothesis (ii) of Lemma 6.1, we estimate

$$|A^{-1}\tilde{F}(q_T,\mu_0)| \leq \tilde{C}(|T_-|^{-2} + e^{-2\lambda^s T_+}) \tag{6.6}$$

using (6.1) and (6.3). Then hypothesis (ii)

$$|A^{-1}\tilde{F}(q_T,\mu_0)| \leq (1-\kappa)\rho$$

reads

$$\tilde{C}(|T_-|^{-2} + e^{-2\lambda^s T_+}) \leq \frac{1}{8C}|T_-|^{-1} \tag{6.7}$$

substituting (6.1) and the expressions for κ and ρ. The last inequality (6.7) holds provided $|T_-|$ is sufficiently large and

$$e^{-2\lambda^s T_+} \leq \frac{1}{C}|T_-|^{-1},$$

for a different constant C, that is, provided

$$C + \frac{1}{2\lambda^s}\ln|T_-| \leq T_+, \tag{6.8}$$

which is precisely the hypothesis made in Theorem 2. Hence, hypothesis (i) of Lemma 6.1 is fulfilled provided T_- and T_+ are chosen according to (6.8) and $|T_-| \geq T_+$ holds. Therefore, the Chord method converges inside the ball $B_\rho(q_T,\mu_0)$ yielding a unique fixed point (u_T,μ_T), which satisfies the estimate

$$|u_T - q_T| + |\mu_T - \mu_0| \leq (1-\kappa)^{-1}|A^{-1}\tilde{F}(q_T,\mu_0)| \leq C(|T_-|^{-2} + e^{-2\lambda^s T_+})$$

by (6.6). This proves part (ii) of Theorem 3.

It remains to prove Theorem 2. To this end, we shall again parametrize along suitable solutions $q^+(c^+)$ and $q^-(c^-)$ contained in the stable and center-unstable manifolds, respectively. These are obtained by applying Lemma 4.1 to $N_1(\mu) = W^s(p(\mu),\mu)$ and $N_2(\mu) = W^{cu}(p(\mu),\mu)$ using the assumption (S3). Hence

$$\begin{array}{lll} q^+(\alpha,b^+,\mu)(t) \in W^s(p(\mu),\mu) & q^+(\alpha,b^+,\mu)(0) = q^+(\alpha,b^+,\mu) & \text{for } t \geq 0 \\ q^-(\alpha,b^-,\mu)(t) \in W^{cu}(p(\mu),\mu) & q^-(\alpha,b^-,\mu)(0) = q^-(\alpha,b^-,\mu) & \text{for } t \leq 0, \end{array}$$

with $c^+ = (\alpha, b^+, \mu)$ and $c^- = (\alpha, b^-, \mu)$. Note that $q^-(c^-)(t) \notin W^u(p(\mu), \mu)$ for all c^-. We consider the linearization

$$\dot{v} = D_u f(q^\pm(c^\pm)(t), \mu)\, v \tag{6.9}$$

along these solutions and denote the evolution of (6.9) by $\Phi_\pm(c^\pm, t, s)$ for either $s, t \geq 0$ or $s, t \leq 0$. Again, (6.9) admits dichotomies.

Lemma 6.2 *For any $\epsilon > 0$ there exists a $\delta > 0$ such that the following holds for $|\mu_0 - \mu| \leq \delta$: there exist projections $P_+^s(c^+)(t)$ and $P_-^{cu}(c^-)(t)$ defined for $t \geq 0$ and $t \leq 0$, respectively, such that*

$$\begin{array}{rcll} |\Phi_+(c^+, t, s) P_+^s(c^+)(s)| & \leq & K e^{-\lambda^s(t-s)} & t \geq s \geq 0 \\ |\Phi_+(c^+, s, t)(1 - P_+^s(c^+)(t))| & \leq & K & t \geq s \geq 0 \\ |\Phi_-(c^-, t, s)(1 - P_-^{cu}(c^-)(s))| & \leq & K e^{-\lambda^s(t-s)} & s \leq t \leq 0 \\ |\Phi_-(c^-, s, t) P_-^{cu}(c^-)(t)| & \leq & K e^{\epsilon|t-s|} & s, t \leq 0 \end{array} \tag{6.10}$$

holds. Denoting the complementary projections by

$$\begin{array}{ll} P_+^{cu}(c^+)(t) := 1 - P_+^s(c^+)(t) & \text{for } t \geq 0 \\ P_-^s(c^-)(t) := 1 - P_-^{cu}(c^-)(t) & \text{for } t \leq 0, \end{array}$$

we have in addition that the subspaces

$$\begin{array}{rcl} \ker P_+^s(c^+)(0) = RP_+^{cu}(c^+)(0) & = & \mathbb{R}\psi(0) \oplus Y^+ \\ \ker P_-^{cu}(c^-)(0) = RP_-^s(c^-)(0) & = & \mathbb{R}\psi(0) \oplus Y^- \end{array} \tag{6.11}$$

do not depend on c^+ or c^-. The estimates in (6.10) are true for the derivatives with respect to c^+ and c^-, too,

We will use the abbreviations

$$\begin{array}{ll} \Phi_+(c^+, t, s) P_+^s(c^+)(s) =: \Phi_+^s(c^+, t, s) & \Phi_+(c^+, s, t) P_+^{cu}(c^+)(t) =: \Phi_+^{cu}(c^+, s, t) \\ \Phi_-(c^-, s, t) P_-^{cu}(c^-)(t) =: \Phi_-^{cu}(c^-, s, t) & \Phi_-(c^-, t, s) P_-^s(c^-)(s) =: \Phi_-^s(c^-, t, s). \end{array}$$

Proof. The statements concerning $\Phi_-(c^-, t, s)$ follow for example from Sandstede (1993, Lemma 1.1). From the same lemma we conclude that $\Phi_+(c^+, t, s)$ possesses stable and unstable dichotomies, which do not include the center direction. It remains to construct a solution of the variational equation along $q^+(c^+)(t)$ which is bounded uniformly on $\mathbb{R}^+$ and depends smoothly on c^+. This can be done as in Sandstede (1993, Lemma 1.5) observing that $|q^+(c^+)(t)| \leq Ce^-\lambda^s t$ converges exponentially for $t \to \infty$, see also Coddington & Levinson (1955, §3.8) or Schecter (1993, Lemma 3.2). □

Again, we use the solutions $q^+(c^+)$ and $q^-(c^-)$ as a parametrization which respect to which solutions of the boundary value problem (3.3) are constructed. Hence, define

$$
\begin{aligned}
u^+(t) &= q^+(c^+)(t) + v^+(t) \qquad && t \geq 0 \\
u^-(t) &= q^-(c^-)(t) + v^-(t) \qquad && t \leq 0.
\end{aligned}
\tag{6.12}
$$

For the same reasoning as in the hyperbolic case, we shall require that

$$
v^+(0) \in \mathbb{R}\psi(0) \oplus Y^+ \qquad v^-(0) \in \mathbb{R}\psi(0) \oplus Y^-,
\tag{6.13}
$$

which are the spaces appearing in (6.11) being independent of c^+ and c^-. Then $u^\pm$ solve the equation

$$
\dot{u} = f(u, \mu)
$$

if and only if the functions $v^\pm$ solve

$$
\dot{v}^\pm = D_u f(q^\pm(c^\pm)(t), \mu)\, v^\pm + g^\pm(t, v^\pm, c^\pm)
\tag{6.14}
$$

for $t \geq 0$ or $t \leq 0$, respectively, for the nonlinearities given by formula (5.7). As in the hyperbolic case, we rewrite (6.14) and (6.13) as an integral equation

$$
\begin{aligned}
v^+(t) &= \Phi_+^{cu}(t, T_+)a^+ + \textstyle\int_{T_+}^t \Phi_+^{cu}(t, s) g^+(t, v^+(s), c^+)\, ds + \\
&\quad \textstyle\int_0^t \Phi_+^s(t, s) g^+(t, v^+(s), c^+)\, ds \\
v^-(t) &= \Phi_-^s(t, T_-)a^- + \textstyle\int_{T_-}^t \Phi_-^s(t, s) g^-(t, v^-(s), c^-)\, ds + \\
&\quad \textstyle\int_0^t \Phi_-^{cu}(t, s) g^-(t, v^-(s), c^-)\, ds,
\end{aligned}
\tag{6.15}
$$

where $a^+ \in E_0^{cu}$ and $a^- \in E_0^s$. We solve (6.15) using the norms

$$
\begin{aligned}
|v|_+ &:= \sup\nolimits_{t \in [0, T_+]} |v(t)| \\
\|v\|_- &:= \sup\nolimits_{t \in [T_-, 0]} e^{-\lambda^s (T_- - t)} |v(t)|
\end{aligned}
$$

in the spaces

$$
\begin{aligned}
V^+ &:= \{v \in C^0([0, T_+], \mathbb{R}^n)\} \\
V^- &:= \{v \in C^0([T_-, 0], \mathbb{R}^n) \,|\, \|v\|_- < \infty\}
\end{aligned}
$$

endowed with the above defined norms. Let $B_\rho^\pm$ denote the ball of radius ρ in the spaces $V^\pm$. Moreover, let

$$
\|a^+\| := |T_+|\, |a^+|
$$

be a norm in $\mathbb{R}^n$.

Lemma 6.3 *There exist $\delta, \rho > 0$ such that for any T_+, $|T_-| > 1/\delta$, any given $(a^+, a^-) \in E_0^{cu} \times E_0^s$ with $\|a^+\| + |a^-| < \delta$, and any $(\alpha, b^+, b^-, \mu) \in \mathbb{R} \times Y^+ \times Y^- \times \mathbb{R}$ with $|\alpha| + |b^+| +$*

$|b^-| + |\mu - \mu_0| < \delta$, *there exist unique solutions* $(v^+, v^-)(a^\pm, c^\pm) \in B^+_{\rho/T_+} \times B^-_\rho$ *of equation (6.15). Moreover,* $v^\pm$ *depends smoothly on* $(a^\pm, c^\pm)$ *and the estimate*

$$|v^+|_+ \leq C\|a^+\| = C|T_+|\,|a^+| \qquad \|v^-\|_- \leq C|a^-|$$

holds for some constant C *depending only on* δ. *The estimate is valid for derivatives with respect to* $c^\pm$*, too.*

Proof. It is easy to see that the right-hand side of (6.15) defines a smooth mapping from $V^\pm$ into itself using the estimate (5.7) for the nonlinearity $g^\pm$, see Lin (1990), Schecter (1995) or Sandstede (1993) for the details. Moreover, we obtain

$$\begin{aligned} |v^+(t)| &\leq KT_+|a^+| + KT_+|v^+|_+^2 \\ \|v^-(t)\| &\leq K|a^-| + K\|v^-\|_-^2, \end{aligned} \tag{6.16}$$

using the estimates (6.10) obtained in Lemma 6.2. Hence, we can employ the implicit function theorem in the ball $B^+_{\rho/T_+} \times B^-_\rho \subset V^+ \times V^-$ for some $\rho > 0$ independent of T_+ or T_-. This proves the lemma. $\square$

Next, we shall consider the boundary conditions at $t = T_+$ and $t = T_-$, respectively, given by

$$\begin{aligned} & P_0^{cu}(\mu)(u(T_+) - p(\mu)) \\ &\quad = P_0^{cu}(\mu)\Big(q^+(c^+)(T_+) + P_+^{cu}(c^+)(T_+)a^+ \\ &\qquad + \int_0^{T_+} \Phi_+^s(T_+, s)g^+(t, v^+(s), c^+)\,ds - p(\mu)\Big) = 0 \end{aligned} \tag{6.17}$$

$$\begin{aligned} & P_0^{s}(\mu)(u(T_-) - p(\mu)) \\ &\quad = P_0^{s}(\mu)\Big(q^-(c^-)(T_-) + P_+^{s}(c^-)(T_-)a^- \\ &\qquad + \int_0^{T_-} \Phi_-^{cu}(T_-, s)g^-(t, v^-(s), c^-)\,ds - p(\mu)\Big) = 0 \end{aligned} \tag{6.18}$$

using (6.15) and the parametrization (6.12).

Lemma 6.4 *For any* T_+, T_- *and* $c^\pm$ *chosen as in Lemma 6.3, there exists a unique pair* $(a^+, a^-) \in E_0^{cu} \times E_0^s$ *such that* $v^\pm(a^\pm(c^\pm), c^\pm)$ *solve the boundary conditions (6.17) and (6.18). Moreover,* (a^+, a^-) *depends smoothly on* (c^+, c^-) *and the estimate*

$$\begin{aligned} |a^+| &\leq C|P_0^{cu}(\mu)(p(\mu) - q^+(c^+)(T_+))| &\leq Ce^{-2\lambda^s T_+} \\ |a^-| &\leq C|P_0^{s}(\mu)(p(\mu) - q^-(c^-)(T_-))| &\leq C\,1/|T_-|^2 \end{aligned} \tag{6.19}$$

holds.

Proof. The maps $P_0^{cu}(\mu)P_+^{cu}(c^+)(T_+)|_{E_0^{cu}}$ and $P_0^s(\mu)P_+^s(c^-)(T_-)|_{E_0^s}$ are isomorphisms for all T_+ and $|T_-|$ sufficiently large as $\lim_{t\to\infty} P_+^{cu}(c^+)(t) = P_0^{cu}(\mu)$ and similarly for $P_-^s(c^-)(t)$, see Palmer (1984, Lemma 3.4) or Sandstede (1993, Lemma 1.1). Using Lemma 5.2 we can therefore solve (6.17) and (6.18) by the implicit function theorem for any fixed T_+ and T_-. The first part of the estimate (6.19) follows now immediately. The second part of the inequalities is a consequence of the quadratic tangency between $W^{cu}(p(\mu),\mu)$ and its tangent space $\mathrm{R}P_0^{cu}(\mu)$ at $p(\mu)$ together with hypothesis (S1) (and an analogous property for $W^s(p(\mu),\mu)$). □

As before, we obtain two pieces of solutions satisfying the differential equation and the boundary conditions at both end points. It remains to choose α, b^+ and b^- such that the pieces match at $t=0$. This follows essentially as in Lemma 5.3 by employing again Lemma 4.2 and hypothesis (S4), whence we omit the proof of the following lemma.

Lemma 6.5 *For any choice of T_+ and T_- as in Lemma 6.3, there exist a unique vector $(\alpha_T, b_T^+, b_T^-, \mu_T)$ such that*

(6.20) $$q^+(\alpha_T, b_T^+, \mu_T)(0) + v^+(\alpha_T, b_T^+, \mu_T)(0) = q^-(\alpha_T, b_T^-, \mu_T)(0) + v^-(\alpha_T, b_T^-, \mu_T)(0)$$

and

$$J_T(u_T, \mu_T) = 0$$

hold. Here,

$$u_T(t) = q^\pm(\alpha_T, b_T^\pm, \mu_T)(t) + v^\pm(\alpha_T, b_T^\pm, \mu_T)(t)$$

for $t \geq 0$ and $t \leq 0$, respectively. Moreover, the estimate

$$|b_T^+| + |b_T^-| + |\mu_T - \mu_0| \leq C(e^{-2\lambda^s T_+} + e^{\lambda^s T_-})$$

hold.

This completes the proof of Theorem 2.

7 Numerical results

Finally, we report on numerical simulations obtained for the computation of saddle-node homoclinic solutions using the integral phase condition and a linear approximation of the center-unstable manifold. The example investigated here is given by

(7.21) $$\begin{aligned} \dot u_1 &= -u_1 - u_2 + u_1^2 + \mu u_1^2(3u_1 - 2) \\ \dot u_2 &= -u_1 - u_2 + 1.5\, u_1^2 + 1.5\, u_1 u_2 + 2\mu u_1 u_2, \end{aligned}$$

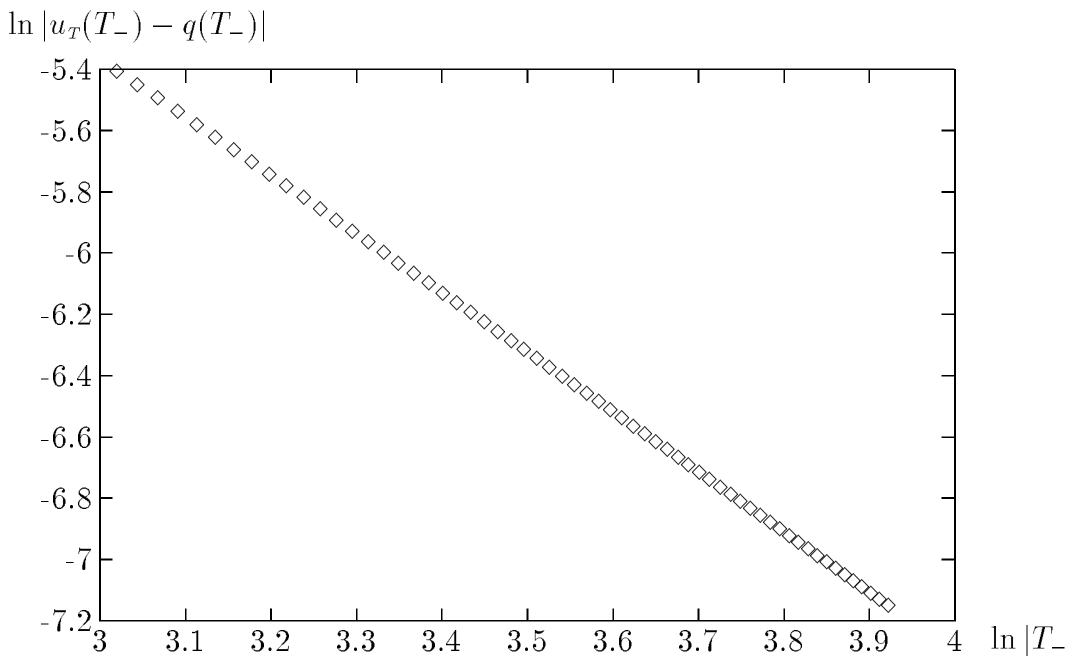

Figure 3: Graph of $\ln|u_T(T_-) - q(T_-)|$ versus $\ln|T_-|$. The slope of the line is equal to -1.94.

see Sandstede (1995). We have the following lemma.

Lemma 7.1 *Equation (7.21) admits a homoclinic solution $q(t) \in W^c(0) \cap W^s(0)$ for $\mu = 0$ satisfying the assumptions (S1) to (S4) stated in section 3 with $\lambda^s = 2$. The homoclinic solution is contained in the cartesian leaf $\{(u_1, u_2) \mid u_2^2 = u_1^2(1-u_1)\}$. Moreover, the center manifold has a non-degenerate quadratic tangency to its tangent space at the equilibrium $p_0 = 0$.*

Proof. The statements follow from Sandstede (1995, Lemma 3.1) and the explicit representation of the center manifold. □

The homoclinic solution $q(t)$ can be computed to arbitrary accuracy by solving the one-dimensional problem on the curve $\gamma(q) = \{(u_1, u_2) \mid u_2^2 = u_1^2(1-u_1),\ u_1 \in (0,1]\}$. The truncated problem is given by

$$
(7.22) \qquad \begin{pmatrix} \dot{u} - f(u,\mu) \\ \int_{T_-}^{T_+} \langle \dot{q}(t), u(t) - q(t) \rangle \, dt \\ u_1(T_+) - u_2(T_+) \\ u_1(T_-) + u_2(T_-) \end{pmatrix} = 0,
$$

for $t \in T = [T_-, T_+]$ and $u = (u_1, u_2)$. The system is solved using a modified version of HomCont, see Champneys, Kuznetsov & Sandstede (1995*b*, 1995*a*), which is a driver for Auto86, see Doedel (1981). The truncation interval is chosen to be $T = [-20.0, 20.0]$ and the homoclinic solution is continued in the parameter T_- ranging from $T_- = -20.0$ to $T_- = -50.0$. From Theorem 2, we expect that the solution u_T of (7.22) satisfies the error estimate

$$|u_T - q_T| \leq K/|T_-|^2.$$

In Figure 3, the endpoint $u_T(T_-)$ is compared to the nearest point $(u_1, u_2) = (x, -x\sqrt{1-x})$ on the cartesian leaf - to which the homoclinic orbit $q(t)$ is constrained - using $x = u_T(T_-)_1$. The slope obtained is equal to -1.94 differing from the expected slope by -0.06.

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